



Tamkang University

Available online at [jims.ms.tku.edu.tw/list.asp](http://jims.ms.tku.edu.tw/list.asp)

International Journal of Information and Management Sciences  
20 (2009), 71-88

---

International Journal of  
**Information**  
and  
**Management**  
**Sciences**

---

[jims.ms.tku.edu.tw](http://jims.ms.tku.edu.tw)

## An Additive Scale Model for the Analytic Hierarchy Process

*Yuh-Yuan Guh*

Department of Business Administration  
Chung Yuan Christian University  
R.O.C.

*Rung-Wei Po*

Institute of Technology Management  
National Tsing Hua University  
R.O.C.

*Kuo-Ren Lou*

Department of Management Sciences & Decision Making  
Tamkang University  
R.O.C.

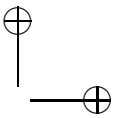
### Abstract

This study presents an additive scale model for the Analytic Hierarchy Process (AHP) that suits the decision problem using a linear preference comparison. This study discusses issues related to mathematical denotation, axiom, transitivity and numerical analysis for the additive scale model of AHP. The least squares method and correlation analysis are used to obtain the relative criteria weights and consistency index. Moreover, a fuzzy model is developed to enhance the practical flexibility of the additive scale model of AHP in applications. Two examples are used to demonstrate that the criteria weights derived from the proposed approach are steady and effectively reflect the intensity of perception, and the consistency index is invariant to the scale multiplier employed.

**Keywords:** Multicriteria, Analytic Hierarchy Process, Pairwise Comparison Scale, Ratio Scale, Additive Scale, Scale Transitivity.

### 1. Introduction

The Analytic Hierarchy Process (AHP) is one of the most used multicriteria decision-making approaches. Proposed by Saaty in the mid 1970s, the AHP combines tangible and intangible features to derive priorities associated with problem alternatives. The AHP uses the well-defined mathematical structure of a reciprocal matrix and the eigenvalue approach to generate true or approximate weights. The AHP is a structural framework



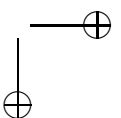
enabling Decision Makers (DMs) to enhance their understanding of complex decisions by dividing the problem into a hierarchical structure. Incorporating all relevant decision criteria, and their pairwise comparisons permits DMs to make trade-offs among objectives. The AHP is suitable for solving complex and challenging evaluation problems; such problems are typical for R&D project selection, investment risk analysis, organizational planning, performance measurement, alternative selection, project evaluation and public policy analysis (Sampson and Showalter [28], Forgionne et al. [9], Kahraman et al. [14], Tsaur et al. [30], Chow and Luk [5], Badri et al. [2], Kahraman et al. [15] and Alkahtani et al. [1]).

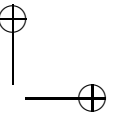
Fundamentally, the AHP provides a "ratio" scale of relative magnitudes expressed in dominance objects to represent judgments in the form of paired comparisons. An overall ratio is then synthesized and used to rank objects, and, thus, ratio transitivity is also implied when deriving the relative weights of objects. In the AHP, ratio transitivity means that if one likes  $A$  twice as much as  $B$  and  $B$  three times as much as  $A$ . If one likes  $C$  six times as much as  $A$ , the transitivity rule is respected. Therefore,  $c(i, j) \in S$  denotes the comparison of objects  $i$  and  $j$ , where  $S$  is the set of possible scale values; the ratio transitivity is then represented as  $c(i, j) = c(i, k) \times c(k, j)$ .

However, according to human perception model in numerous decision-making situations, the ratio scale is clearly not the only frame of reference used to describe preferability between two objects. For example, one might say that they like  $A$  two points (not twice) as much as  $B$ . This scenario implies that in some cases, it is intuitively reasonable to refer to an anchor point and use the interval (distance) of two objects to express relative preferability. Of course, additive transitivity is implied in these cases; namely, if one likes  $A$  two points better than  $B$ , and  $B$  three points better than  $C$ , one should like  $A$  five points better than  $C$ . Harker and Vargas [12], Lootsma [20], Torgerson [31], Ali et al. [39] and Hochbaum [40] stated that additive sense is another perception model. Additive transitivity is represented as  $c(i, j) = c(i, k) + c(k, j)$ .

Due to the conventional AHP models developed all based on ratio scale with transitivity, additive sense is also the human perception model for pairwise comparison. Instead of a ratio scale, this study proposes an additive scale for the AHP model, which is based on additive transitivity, and further developed a consistency index (CI) to monitor whether the DM is consistent or rational when performing interval comparisons among a cluster of objects.

The remainder of this paper is organized as follows. First, the basic concepts of the pairwise comparison scale and ratio model of the AHP are briefly presented. The additive scale model designed for the AHP is then demonstrated, including the additive scale of the axiom, verbal and numerical components. The least square method is then used to obtain the relative estimated weights for a cluster of criteria, and the correlation coefficient is employed to develop a CI for the additive AHP. Furthermore, several cases are implemented to test whether the proposed model is a valid and acceptable method for eliciting and analyzing subjective judgments. Finally, a fuzzy model is developed to augment the practical flexibility of the additive scale for the AHP model in application.





## 2. Pairwise Comparison Scale

Decision making, especially regarding intangible stimuli or criteria, such as degree of quality or attitude factors, is a very hard task. Not only is information about stimuli often imprecise or incomplete, DM judgment is sometimes inconsistent. Given a cluster of related stimuli or criteria, one way to evaluate their relative intensity of preference (or importance) is to perform pairwise comparisons. Hence, the pairwise comparison approach is widely used. Belton and Gear [3], Belton and Gear [4], Foster and Al-Dubaibi [11] and Saaty [24] indicated that the comparison scale provides a decision model that aids DMs in making judgments by stating the degree to which one object is preferred over another. The “how much” answer is given by a DM examining a preset scale and identifying the response on that scale that most closely approximates felt response.

The comparison scale is therefore a crucial factor for eliciting DM responses, and producing an objective evaluation. Torgerson [31] stated that the main comparison scale has multiplicative and additive comparison scales.

Multiplicative scale :  $c(i, j) = 1/c(j, i)$

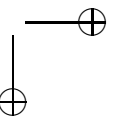
Additive scale :  $c(i, j) = -c(j, i)$

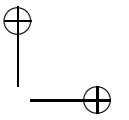
According to ratio transitivity,  $c(i, j) = c(i, k) \times c(k, j)$ , due to  $c(i, i) = c(i, j) \times c(j, i) = 1$ , the multiplicative scale  $c(i, j) = 1/c(j, i)$  is thus derived. It means that ratio transitivity is a necessary condition for a multiplicative scale. Similarly, according to additive transitivity,  $c(i, j) = c(i, k) + c(k, j)$ , due to  $c(i, i) = c(i, j) + c(j, i) = 0$ , the additive scale  $c(i, j) = -c(j, i)$  is derived, meaning that additive transitivity is a necessary condition for an additive scale. Multiplicative scale is the so-called “ratio scale,” whereas additive scale is called an “interval scale.”

The scale used has a significant effect on outcome consistency and accuracy. Thus, issues related to scales or scaling systems have drawn considerable attention. Ji and Jiang [13] proposed an analytical structure for existing scales by decomposing a scale into verbal and numerical parts. A well-defined verbal part of a scale should include the following three components: (1) number of relative importance gradations; (2) semantic definition for each gradation; and, (3) relationships among different gradations. This study uses the above analytical structure to analyze the additive scale designed for the AHP model.

## 3. Ratio Scale of the AHP

The AHP is primarily used for resolving of choice problems in a multi-criteria environment. The AHP converts individual preferences into ratio-scale weights that are combined into linear additive weights for the associated alternatives. When implementing the AHP process, following hierarchy construction, the objects within each cluster and of each cluster within the group of clusters are evaluated using pairwise comparisons. There are  $n(n-1)/2$  judgments are made regarding the relative importance for a decision





problem involving  $n$  criteria, and a square matrix structure is eventually established via these pairwise comparisons.

Let  $c_1, c_2, c_3, \dots, c_n$  be the set of criteria to be compared, with weights denoted as  $w_1, w_2, w_3 \dots$  and  $w_n$ . The objective of AHP is to estimate the relative weights of criteria  $c_i, i = 1, \dots, n$ , when a series of pairwise ratio comparisons are performed for  $w_i$  and  $w_j, i, j = 1, \dots, n$ , this decision problem can be generalized as an  $n \times n$  square matrix  $C$ .  $P(c_i, c_j) = a_{ij}$ .

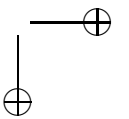
$$C = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_{n-1} & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_{n-1} & w_2/w_n \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ w_{n-1}/w_1 & w_{n-1}/w_2 & \cdots & w_{n-1}/w_{n-1} & w_{n-1}/w_n \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_{n-1} & w_n/w_n \end{bmatrix}. \quad (1)$$

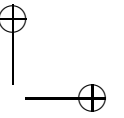
While  $a_{ij}$  represents a relative importance ratio judged by DM for each pair of  $w_i/w_j$ , this yields an  $n$ -by- $n$  matrix  $A$ , where  $P(c_i, c_i) = a_{ii} = 1$  and  $P(c_j, c_i) = a_{ji} = 1/a_{ij}, i, j = 1, \dots, n$ . Matrix  $A$  is a positive reciprocal pairwise comparison matrix, while  $A$  is the consistency matrix, the relationship between  $w_i, w_j$  and judgment  $a_{ij}$  are simply given  $w_i/w_j = a_{ij}$ , for  $i, j = 1, \dots, n$ .

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}. \quad (2)$$

Vargas [32] stated that the AHP approach does not adhere to the conventional Multi-attribute utility theory (MAUT) axiom of transitivity. The goal of MAUT is to find a simple expression for DMs' preference. Unlike MAUT, AHP uses a quantitative comparison method that is based on pairwise comparison of decision criteria, rather than utility and weighting functions. The MAUT relies on the assumptions that the DM is rational, preferences do not change, and the DM has perfect knowledge and makes consistent judgments. The AHP technique relies on the supposition that humans are more capable of making relative judgments than absolute judgments. Therefore, the rationality assumption in the AHP is more relaxed than that in MAUT, permitting the input value of comparison between criteria to be intransitive.

Crawford and Williams [34], Golany and Kress [35], Jensen [36], Saaty and Vargas [37] and Takeda et al. [38] compared the possible tools for deriving relative weights from a pairwise comparison matrix, including Least Squares Method, Logarithmic Least Squares Method, Weighted Least Squares Method, Chi Squares Method, Logarithmic Least Absolute Values Method and Singular Value Decomposition. Moreover, to correct this intransitivity, the consistency index ( $CI$ ) and consistency ratio ( $CR$ ) are developed to assess the consistency of comparison matrix  $A$ . The  $CI$  and  $CR$  are calculated as





$CI = (\lambda_{\max} - n)/(n - 1)$ ,  $CR = CI/RI$ , where  $RI$  represents the average  $CI$  over numerous random entries of the same order reciprocal matrices, and  $\lambda_{\max}$  is the largest eigenvalue of matrix  $A$ .

To build up the foundation of AHP theories, Saaty [25] showed the ratio scale model of AHP is founded on the following set of axioms for deriving a scale from fundamental measurements and for hierarchical composition, (1) Reciprocal axiom: If criterion  $c_i$  is  $P(c_i, c_j)$  times more important than criterion  $c_j$ , then criterion  $c_j$  is  $1/P(c_i, c_j)$  times as important than criterion  $c_i$ . (2) Homogeneity axiom: Only comparable elements are compared. It is essential for comparing similar things, as judgment errors become large when comparing very disparate elements. (3) Independence axiom: The relative importance of elements at any level does not depend on what elements are included at a lower level. (4) Expectation axiom: The hierarchy must be complete and include all the criteria and alternatives in the subject under study. No criteria and alternatives are left out and no excess criteria and alternatives are included.

The design of comparison scale gradations is also a necessary for pairwise comparison. Given a stimulus or object in an ideal situation with exact positive gradation  $v_1, v_2, \dots, v_m$  is assigned to stimuli. In the ratio scale model of the AHP, numerical transitivity among gradation scales is  $v_k v_j = v_{k \times j}$ , where  $k, j = 1, 2, \dots, m$  and  $k + j - 1 \leq 9$ . The most cited ratio model of the AHP is the Saaty scale, which includes 1-9 gradations with the following five major semantic grades: 1-equal (equal important), 3-moderate (slightly more important), 5-strong (strongly more important), 7-very strong (demonstrably more important) and 9-absolute (absolutely more important). The Saaty scale also has four threshold gradations as intermediate states between the two adjacent major gradations. Foster & Al-Dubaibi [11] indicated that the Saaty developed the 1-9 gradation, using a geometric series of stimuli based on the psycho-physical law of Weber and Fechner, and because individuals cannot simultaneously compare more than  $7 \pm 2$  objects without becoming confused.

As the scale used with its transitivity among scale gradation has a significant effect on the AHP outcome's consistency and accuracy; numerous scholars had developed different scales on ratio comparison. Ji & Jiang [13] denoted the numerical part of ratio scale as  $\{v_i, 1, 1/v_i\}$  and reviewed five main scales for AHP as follows:

(1) Saaty scale[23]

$$v_i = i, m = 9; \text{ that is, } 1, 2, 3 \dots 9$$

(2) Ma-Zheng scale[22]

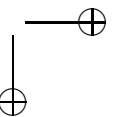
$$v_i = \frac{9}{10 - i}, m = 9; \text{ that is, } 1, 9/8, 9/7 \dots 9$$

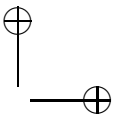
(3) Donegan-Dodd-McMaster scale[6, 7]

$$v_i = \exp \left[ \tanh^{-1} \left( \frac{i - 1}{H - 1} \right) \right], H = 1 + 6/\sqrt{2} \text{ or } 1 + 14/\sqrt{3}, m = 9$$

(4) Lootsma scale [21]

$$v_i = c^{i-1}, c = \sqrt{2} \text{ or } 2, m = 7 \text{ or } 9$$





(5) Salo-Hamalainen scale or the balanced scale [27]

$$v_i = \frac{0.5 + (i-1)s}{0.5 - (i-1)s}, \quad s = 0.05 \text{ or } 1/17, \quad m = 9$$

Each of these scales has a different effect on scale transitivity. For instance, Satty scale,  $v_{AB} = v_3$  and  $v_{BC} = v_3$ , then  $v_{AC} = v_{AB} \times v_{BC} = v_9$  ("A is moderately important than B and B is moderately important than C; then A should be absolutely more important than C"). Belton and Gear [3] and Forman and Gass [10] queried that the Saaty scale transits too rapid for some practical applications, and thus Dodd et al. [6] developed the Donegan-Dodd-McMaster scale, which is known as 8-based horizon  $H = 1 + 14/\sqrt{3}$  based on the assumption that if  $v_{AB} = v_{BC} = v_8$ , then  $v_{AC} = v_{AB} \times v_{BC} = v_9$  (namely,  $8 \times 8 = 9$ ), and the 7-based horizon  $H = 1 + 6/\sqrt{2}$  based on the assumption that if  $v_{AB} = v_{BC} = v_7$ , then  $v_{AC} = v_{AB} \times v_{BC} = v_7 \times v_7 = v_9$  (namely,  $7 \times 7 = 9$ ). Therefore, the Donegan-Dodd-McMaster scale has a moderate effect on ratio scale transitivity, smaller than that of the Saaty scale.

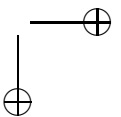
In the following sections, present an additive scale model for the AHP to suit the decision environment using a linear preference comparison. The proposed additive scale of AHP model also has a moderate effect on scale transitivity, smaller than that of the Saaty scale.

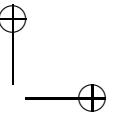
#### 4. Additive Scale of the AHP

As stated conventional AHP models adopt the ratio scale; however, ratio transitivity is not only way to represent and evaluate preferability among objects. According to the people's perception model in evaluation, using interval (distance) to express their relative preferability between two objects is a natural behaviour. Hence a pairwise subtraction comparison is made for criteria  $c_i$  and  $c_j$ ,  $i, j = 1, \dots, n$ . This decision problem thus can be generalized as an  $n \times n$  square matrix  $C'$ .

$$C' = \begin{bmatrix} w_1 - w_1 & w_1 - w_2 & \cdots & w_1 - w_{n-1} & w_1 - w_n \\ w_2 - w_1 & w_2 - w_2 & \cdots & w_2 - w_{n-1} & w_2 - w_n \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ w_{n-1} - w_1 & w_{n-1} - w_2 & \cdots & w_{n-1} - w_{n-1} & w_{n-1} - w_n \\ w_n - w_1 & w_n - w_2 & \cdots & w_n - w_{n-1} & w_n - w_n \end{bmatrix}. \quad (3)$$

This study defines  $d_{i,j}$  as the interval comparison made by a DM in comparing criterion  $c_i$  with criterion  $c_j$ , and yields an  $n \times n$  matrix  $D$  ( $d_{ij} = -d_{ji}$ ,  $d_{ii} = 0$ , for  $i, j = 1, \dots, n$ ).  $D$  is a skew-symmetric matrix (or antisymmetric) with a transpose that is also its negative. Although skew-symmetry is a necessary condition for the consistency of a rankings matrix, it is insufficient. If  $D$  is the consistency matrix, the relationship





between  $w_i - w_j$  equals judgment  $d_{i,j}$  for  $i, j = 1, \dots, n$ .

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1,n-1} & d_{1,n} \\ d_{21} & d_{22} & \cdots & d_{2,n-1} & d_{2,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,n-1} & d_{n-1,n} \\ d_{n,1} & d_{n,2} & \cdots & d_{n,n-1} & d_{n,n} \end{bmatrix}. \quad (4)$$

This is the model form for additive type of the AHP. Underlying the established  $C'$  matrix and  $D$  matrix, this study evaluates  $w_i$ ,  $i = 1, \dots, n$ , and ensures that these weights satisfy the scale transitivity property (namely  $w_i - w_j = (w_i - w_k) + (w_k - w_j)$ ).

The proposed additive scale model is a new approach for the AHP methodology. Establishing an axiomatic foundation is necessary for deriving theorems. Here, this study revises the axioms founded by Saaty [25] for the ratio scale model of AHP (listed in Section 3). As implementation of the proposed model is also based on a hierarchy evaluation structure, the Homogeneity, Independence, and Expectation axioms mentioned in Section 3 are still applicable except the Reciprocal axiom. Additionally, this study defines the Subtraction axiom for additive scale of AHP model instead of the Reciprocal axiom in ratio scale of AHP model. The equation is defined as follows:

$$P(c_i, c_j) = -P(c_j, c_i), \quad \forall c_i, c_j \in C \quad (5)$$

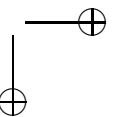
To design a comparison scale gradations according to the psycho-physical law of Weber and Fechner, this study uses the 0–10 gradations to represent preferability for additive scale of AHP model, and thus the comparison scale is in the range of  $[-10, 10]$ . The positive gradation system thus has eleven grades (0, 1, 2 ... 10) and an increment of 1. Thus six major semantic gradations (Table 1) and four threshold gradations are defined as intermediate states between the two adjacent major gradations.

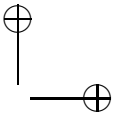
Compared to the ratio AHP model, the additive AHP is numerically denoted as:

$$\{v_i, 0, -v_i\}, \quad v_i = i, \quad i = 0, 1, 2, 3, \dots, 10$$

**Table 1.** Semantics of additive scale for AHP.

Grade	Semantics
0	Equal (equal important)
1	Slightly (slightly more important)
3	Moderate (more important)
5	Strong (strongly more important)
7	Very strong (demonstrably more important)
9	Extremely strong (extremely more important)
10	Absolutely strong (absolutely more important)
2, 4, 6, 8	Compromises/between





The scale structure of the additive AHP model is completely linear. The scale transitivity among different gradations of this model is  $v_k v_j = v_{k+j}$ , where  $k, j = 0, 1, 2, \dots, 10$  and  $k + j - 1 \leq 10$ . For example, the semantics of  $v_3 v_3 = v_6$  is if  $A$  is moderate important than  $B$  and  $B$  is also moderate important than  $C$ , then  $A$  is between strong and very strong important than  $C$ . It is obvious that the scale transitivity is relative milder than the same example in the ratio scale model ( $v_3 v_3 = v_9$ ) and reasonable for practical applications.

## 5. Computing for the Weights and Consistency Index

This section use the least squares method to derive the relative importance weights  $w_i$  for criterion  $c_i$ . Additionally, the Pearson or product-moment correlation is utilized to generate the  $CI$  for additive AHP model.

Let  $\hat{w}_{i,j} = \hat{w}_i - \hat{w}_j$  be the estimator of  $d_{i,j}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, n$ . In other words, this study estimates  $w_i$  such that the sum of the squares of the difference between the  $d_{i,j}$  and its estimator  $w_i - w_j$  is minimized. The sum of the squares difference of  $d_{i,j}$  and  $w_i - w_j$ ,  $i, j = 1, \dots, n$  is:

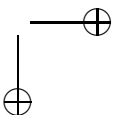
$$\begin{aligned}
 S &= \sum_{i=1}^n \sum_{j=1}^n (\hat{w}_{i,j} - d_{i,j})^2 \\
 &= \sum_{i=1, i \neq k}^n \sum_{j=1, j \neq k}^n (\hat{w}_{i,j} - d_{i,j})^2 + \sum_{i=1, i \neq k}^n (\hat{w}_{i,k} - d_{i,k})^2 + \sum_{j=1, j \neq k}^n (\hat{w}_{k,j} - d_{k,j})^2 + (\hat{w}_{k,k} - d_{k,k})^2 \\
 &= \sum_{i=1, i \neq k}^n \sum_{j=1, j \neq k}^n (\hat{w}_i - \hat{w}_j - d_{i,j})^2 + \sum_{i=1, i \neq k}^n (\hat{w}_i - \hat{w}_k - d_{i,k})^2 + \sum_{j=1, j \neq k}^n (\hat{w}_k - \hat{w}_j - d_{k,j})^2 \\
 &\quad (\because \hat{w}_{k,k} = d_{k,k} = 0) \\
 &\quad (\because (\hat{w}_i - \hat{w}_k) = -(\hat{w}_k - \hat{w}_i), d_{i,k} = -d_{k,i}, (\hat{w}_i - \hat{w}_k - d_{i,k}) = -(\hat{w}_k - \hat{w}_i - d_{k,i})) \\
 &= \sum_{i=1, j \neq k}^n \sum_{j=1, j \neq k}^n (\hat{w}_i - \hat{w}_j - d_{i,j})^2 + 2 \sum_{j=1, j \neq k}^n (\hat{w}_k - \hat{w}_j - d_{k,j})^2 \tag{6}
 \end{aligned}$$

The least squares estimator of  $w_k$ , say  $\hat{w}_k$ , must satisfy  $\frac{\partial S}{\partial \hat{w}_k} = 0$ , this leads to the following equations:

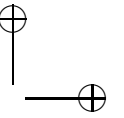
$$\sum_{j=1, j \neq k}^n (\hat{w}_k - \hat{w}_j - d_{k,j}) = 0 \tag{7}$$

$$\Rightarrow \sum_{j=1, j \neq k}^n \hat{w}_k - \sum_{j=1, j \neq k}^n \hat{w}_j - \sum_{j=1, j \neq k}^n d_{k,j} = 0 \tag{8}$$

$$\Rightarrow \sum_{j=1, j \neq k}^n \hat{w}_k + \hat{w}_k - \sum_{j=1, j \neq k}^n \hat{w}_j - \hat{w}_k - \sum_{j=1, j \neq k}^n d_{k,j} - d_{k,k} = 0 \tag{9}$$







$$\Rightarrow \sum_{j=1}^n \hat{w}_k - \sum_{j=1}^n \hat{w}_j - \sum_{j=1}^n d_{k,j} = 0 \quad (10)$$

$$\Rightarrow n\hat{w}_k = \sum_{j=1}^n \hat{w}_j + \sum_{j=1}^n d_{k,j} \quad (11)$$

$$\Rightarrow \hat{w}_k = \left( M + \sum_{j=1}^n d_{k,j} \right) / n \quad \left( \text{let } \sum_{j=1}^n \hat{w}_j = M \right). \quad (12)$$

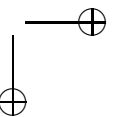
To keep  $w_k \geq 0$ ,  $M + \sum_{j=1}^n d_{k,j}$  must be greater than or equal to zero. Due to the minimum value of  $\sum_{j=1}^n d_{k,j} = \sum_{j=1}^n \hat{w}_k - \sum_{j=1}^n \hat{w}_j$  being  $-10(n-1)$ ,  $M$  is assigned the value  $10(n-1)$ . The estimated weight of  $\hat{w}_k$  criterion  $c_k$  is  $(10(n-1) + \sum_{j=1}^n d_{k,j})/n \geq 0$ . Since  $\sum_{k=1}^n \hat{w}_k = \sum_{k=1}^n (10(n-1) + \sum_{j=1}^n d_{k,j})/n = \sum_{k=1}^n (10(n-1)/n) + \sum_{k=1}^n \sum_{j=1}^n (d_{k,j}/n) = 10(n-1)$ , a normalized weight  $\hat{w}'_k = \hat{w}_k/10n(n-1)$  is taken for  $\hat{w}_k$  to ensure  $\sum_{k=1}^n \hat{w}'_k = 1$ .

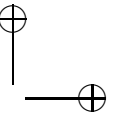
Suppose  $\hat{w}_r$ ,  $\hat{w}_s$  and  $\hat{w}_t$  are three criteria weights derived from the additive AHP model, these criteria are estimated by  $(10(n-1) + \sum_{j=1}^n d_{r,j})/n$ ,  $(10(n-1) + \sum_{j=1}^n d_{s,j})/n$  and  $(10(n-1) + \sum_{j=1}^n d_{t,j})/n$ . The derived weight satisfying additive transitivity is proved as following:

$$\begin{aligned} (\hat{w}_r - \hat{w}_s) + (\hat{w}_s - \hat{w}_t) &= \left( 10(n-1) + \sum_{j=1}^n d_{r,j} \right) / n - \left( 10(n-1) + \sum_{j=1}^n d_{s,j} \right) / n \\ &\quad + \left( 10(n-1) + \sum_{j=1}^n d_{s,j} \right) / n - \left( 10(n-1) + \sum_{j=1}^n d_{t,j} \right) / n \\ &= \left( 10(n-1) + \sum_{j=1}^n d_{r,j} \right) / n - \left( 10(n-1) + \sum_{j=1}^n d_{t,j} \right) / n \\ &= (\hat{w}_r - \hat{w}_t) \end{aligned} \quad (13)$$

Similar to the ratio model of AHP, the additive AHP model also allows the input value of comparison between objects to be intransitive. A *CI*, which is required to assess the consistency of additive transitivity in subjective decisions made by a DM, is formulated as follows:

$d_{i,j}$  and  $\hat{w}_i - \hat{w}_j$  are two measures of the preferability of criterion  $c_j$  with respect to criterion  $c_i$ .  $d_{i,j}$  is the numerical assignment made by a DM, which may not satisfy additive transitivity, whereas  $\hat{w}_i - \hat{w}_j$  is the estimated value derived by using the least squares method, which does satisfy additive transitivity. Thus, the *CI* for the additive





AHP model can be derived using the Pearson or product-moment correlation of pair series  $\hat{w}_i - \hat{w}_j$  and  $d_{i,j}$ ,  $i, j = 1, \dots, n$ .

$$\begin{aligned}
 CI &= \frac{S_{\hat{w}d}}{S_{\hat{w}}S_d} = \frac{\sum_{i=1}^n \sum_{j=1}^n (\hat{w}_{i,j} - \bar{\hat{w}}_{i,j})(d_{i,j} - \bar{d}_{i,j})}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n (\hat{w}_{i,j} - \bar{\hat{w}}_{i,j})^2} \sqrt{\sum_{i=1}^n \sum_{j=1}^n (d_{i,j} - \bar{d}_{i,j})^2}} \\
 &= \frac{\sum_{i=1}^n \sum_{j=1}^n (\hat{w}_{i,j})(d_{i,j})}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n (\hat{w}_{i,j})^2} \sqrt{\sum_{i=1}^n \sum_{j=1}^n (d_{i,j})^2}} \quad (\because \bar{\hat{w}}_{i,j} = 0, \bar{d}_{i,j} = 0) \\
 &= \frac{\sum_{i=1}^n \sum_{j=1}^n (\hat{w}_i - \hat{w}_j)(d_{i,j})}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n (\hat{w}_i - \hat{w}_j)^2} \sqrt{\sum_{i=1}^n \sum_{j=1}^n (d_{i,j})^2}} \\
 &= \frac{\sum_{i=1}^n \sum_{j=1}^n \left( \sum_{k=1}^n d_{i,k} - \sum_{k=1}^n d_{j,k} \right) (d_{i,j})}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n \left( \sum_{k=1}^n d_{i,k} - \sum_{k=1}^n d_{j,k} \right)^2} \sqrt{\sum_{i=1}^n \sum_{j=1}^n (d_{i,j})^2}} \quad \left( \because \hat{w}_i = \left( \sum_{j=1}^n d_{i,j} + M \right) / n \right) \quad (14)
 \end{aligned}$$

The  $CI$  for the additive AHP model ranges from  $-1.0$  to  $+1.0$ . When the  $CI$  is nearly  $+1$ , the judgments made by the DM are consistent. However, when the index is below  $0$ , the  $d_{i,j}$  assignment made by a DM is inconsistent; thus the  $d_{i,j}$  needs to be reassigned by the DM.

Moreover, when  $d_{i,j}$  is enlarged by a multiplier  $k$  to  $kd_{i,j}$ ,  $i, j = 1, \dots, n$ , the correlation coefficient of pair series  $\hat{w}_i - \hat{w}_j$  and  $kd_{i,j}$ ,  $i, j = 1, \dots, n$  is the same as that of the pair series  $\hat{w}_i - \hat{w}_j$  and  $d_{i,j}$ ,  $i, j = 1, \dots, n$ . This means that the consistency index for the additive AHP model is invariant to the scale multiplier.

### Example 1.

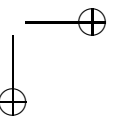
The following instances are used to verify the feasibility of the proposed additive AHP model.

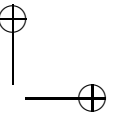
**Case 1.** Comparison between the extreme and moderate cases.

$$D_1 = \begin{bmatrix} 0 & 10 & 10 & 10 \\ -10 & 0 & 10 & 10 \\ -10 & -10 & 0 & 10 \\ -10 & -10 & -10 & 0 \end{bmatrix}, \quad C'_1 = \begin{bmatrix} 0 & 5 & 10 & 15 \\ -5 & 0 & 5 & 10 \\ -10 & -5 & 0 & 5 \\ -15 & -10 & -5 & 0 \end{bmatrix}$$

$$CI = 0.9128$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.500, 0.333, 0.167, 0.000)$$





$$D_2 = \begin{bmatrix} 0 & 5 & 5 & 5 \\ -5 & 0 & 5 & 5 \\ -5 & -5 & 0 & 5 \\ -5 & -5 & -5 & 0 \end{bmatrix}, C'_2 = \begin{bmatrix} 0 & 10/3 & 20/3 & 30/3 \\ -10/3 & 0 & 10/3 & 20/3 \\ -20/3 & -10/3 & 0 & 10/3 \\ -30/3 & -20/3 & -10/3 & 0 \end{bmatrix}$$

$$CI = 0.9128$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.375, 0.292, 0.208, 0.125)$$

Matrix  $D_1$  is an extreme instance, and criterion  $c_1$  is perceived as absolutely more important than other criteria and assigned the largest scale of 10 to  $d_{1,2}$ ,  $d_{1,3}$  and  $d_{1,4}$ . Contrary to criterion  $c_1$ , criterion  $c_4$  is perceived as absolutely unimportant and assigned an extreme scale of  $-10$  to  $d_{4,1}$ ,  $d_{4,2}$ ,  $d_{4,3}$ . The extreme instance shows that the normalized estimated criteria weights  $\hat{w}'_1 = 0.5$  and  $\hat{w}'_4 = 0$  are significantly different; the other  $\hat{w}'_2$  and  $\hat{w}'_3$  are 0.333 and 0.167 respectively.

Matrix  $D_2$  involves a similar case to Matrix  $D_1$ , with the perceived order of importance among criteria the same as that in  $D_1$ , but with the intensity being only half of that in  $D_1$  (scale-5). The normalized estimated criteria weights are  $\hat{w}'_1 = 0.375$ ,  $\hat{w}'_2 = 0.292$ ,  $\hat{w}'_3 = 0.208$  and  $\hat{w}'_4 = 0.125$ . It is obvious that the estimated weights of these criteria are relatively uniform than those in  $D_1$ .

Case 1 indicates that the additive AHP model can effectively reflect the intensity of perception of criteria weights. Notably, instances  $D_1$  and  $D_2$  share the same  $CI$  ( $CI = 0.9128$ ), implying that while the scale magnitude used by DMs may differ, the  $CI$  is not affected; that is,  $CI$  of the additive AHP model is invariant to the scale multiplier used.

**Case 2.** A sensitive analysis for additive model of AHP.

$$D_3 = \begin{bmatrix} 0 & 2 & -2 & 4 \\ -2 & 0 & -4 & 2 \\ 2 & 4 & 0 & 6 \\ -4 & -2 & -6 & 0 \end{bmatrix}, C'_3 = \begin{bmatrix} 0 & 2 & -2 & 4 \\ -2 & 0 & -4 & 2 \\ 2 & 4 & 0 & 6 \\ -4 & -2 & -6 & 0 \end{bmatrix}$$

$$CI = 1.0000$$

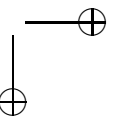
$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.283, 0.217, 0.350, 0.150)$$

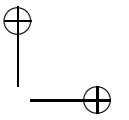
$$D_4 = \begin{bmatrix} 0 & 2 & -2 & 4 \\ -2 & 0 & -4 & 2 \\ 2 & 4 & 0 & 2 \\ -4 & -2 & -2 & 0 \end{bmatrix}, C'_4 = \begin{bmatrix} 0 & 2 & -1 & 3 \\ -2 & 0 & -3 & 1 \\ 1 & 3 & 0 & 4 \\ -3 & -1 & -4 & 0 \end{bmatrix}$$

$$CI = 0.9128$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.283, 0.217, 0.317, 0.183)$$

$$D_5 = \begin{bmatrix} 0 & 2 & -2 & 4 \\ -2 & 0 & -4 & 2 \\ 2 & 4 & 0 & 0 \\ -4 & -2 & 0 & 0 \end{bmatrix}, C'_5 = \begin{bmatrix} 0 & 2 & -2/4 & 10/4 \\ -2 & 0 & -10/4 & 2/4 \\ 2/4 & 10/4 & 0 & 12/4 \\ -10/4 & -2/4 & -12/4 & 0 \end{bmatrix}$$





$$CI = 0.7687$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.283, 0.217, 0.292, 0.208)$$

$$D_6 = \begin{bmatrix} 0 & 2 & -2 & 4 \\ -2 & 0 & -4 & 2 \\ 2 & 4 & 0 & -2 \\ -4 & -2 & 2 & 0 \end{bmatrix}, C'_6 = \begin{bmatrix} 0 & 2 & 0 & 2 \\ -2 & 0 & -2 & 0 \\ 0 & 2 & 0 & 2 \\ -2 & 0 & -2 & 0 \end{bmatrix}$$

$$CI = 0.5773$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.283, 0.217, 0.267, 0.217)$$

$$D_7 = \begin{bmatrix} 0 & 2 & -2 & 4 \\ -2 & 0 & -4 & 2 \\ 2 & 4 & 0 & -4 \\ -4 & -2 & 4 & 0 \end{bmatrix}, C'_7 = \begin{bmatrix} 0 & 2 & 2/4 & 6/4 \\ -2 & 0 & -6/4 & -2/4 \\ -2/4 & 6/4 & 0 & 4/4 \\ -6/4 & 2/4 & -4/4 & 0 \end{bmatrix}$$

$$CI = 0.4082$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.283, 0.217, 0.258, 0.242)$$

$$D_8 = \begin{bmatrix} 0 & 2 & -2 & 4 \\ -2 & 0 & -4 & 2 \\ 2 & 4 & 0 & -6 \\ -4 & -2 & 6 & 0 \end{bmatrix}, C'_8 = \begin{bmatrix} 0 & 2 & 1 & 1 \\ -2 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

$$CI = 0.3162$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.283, 0.217, 0.250, 0.250)$$

Case 2 is a sensitivity analysis for the additive AHP model based on increasing  $d_{3,4}$  to test the steadiness of the  $CI$  and the normalized estimated criteria weights. From instances  $D_4$  to  $D_7$ , the preference intensity of criterion  $c_3$  to criterion  $c_4$  continues reducing (the decrement of  $d_{3,4}$  is 2), and the resulting weight gap between criteria  $c_3$  and  $c_4$  reduces synchronously. The normalized estimated criteria weights  $\hat{w}'_3$  decreases from 0.350 to 0.250, increases  $\hat{w}'_4$  from 0.183 to 0.250. Moreover, the  $CI$  varies steadily from  $D_3$  instance ( $CI = 1.000$ ) to  $D_7$  instance ( $CI = 0.3162$ ).

Illustrative instances in Case 2 demonstrate, in the additive AHP model, that a small changes in elements of comparison matrix do not cause large changes in the estimated criteria weights, thereby satisfying the statement that the steadiness is a necessary requirement for a “goodness” model. (Fichtner [8])

**Case 3.** Comparison of ratio scale and additive model of AHP.

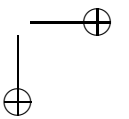
**additive scale model of AHP**

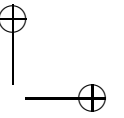
$$D_9 = \begin{bmatrix} 0 & 10 & 10 & 10 \\ -10 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 \end{bmatrix},$$

**ratio scale model of AHP**

$$A_9 = \begin{bmatrix} 1 & 10 & 10 & 10 \\ 1/10 & 1 & 1 & 1 \\ 1/10 & 1 & 1 & 1 \\ 1/10 & 1 & 1 & 1 \end{bmatrix}$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.5, 0.167, 0.167, 0.167) \quad (\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.769, 0.0769, 0.0769, 0.0769)$$





$$D_{10} = \begin{bmatrix} 0 & 10 & 10 & 10 \\ -10 & 0 & 10 & 10 \\ -10 & -10 & 0 & 10 \\ -10 & -10 & -10 & 0 \end{bmatrix}, \quad A_{10} = \begin{bmatrix} 1 & 10 & 10 & 10 \\ 1/10 & 1 & 10 & 10 \\ 1/10 & 1/10 & 1 & 1 \\ 1/10 & 1/10 & 1/10 & 1 \end{bmatrix}$$

$$(\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.5, 0.333, 0.167, 0) \quad (\hat{w}'_1, \hat{w}'_2, \hat{w}'_3, \hat{w}'_4) = (0.691, 0.218, 0.069, 0.022)$$

Next consider the illustrative instances in Case 3. Instance  $D_9$  shows that the estimated criteria weights derived from the additive AHP model, using the linear preference comparison, are relatively uniform than those derived from instance  $A_9$  which is a ratio scale model of the AHP.

Moreover, the instance  $D_{10}$  demonstrates the concept and mechanism of criteria weights derived by the additive AHP model. In instance  $D_{10}$ , the intensity of preferences among criterion  $c_i$  is a gradation structure (that is, criterion  $c_1$  is absolutely preferable to criteria  $c_2, c_3$  and  $c_4$ ; criterion  $c_2$  is absolutely preferable to criteria  $c_3$  and  $c_4$ ; criterion  $c_3$  is absolutely preferable to criterion  $c_4$ ), and, thus the estimated weights of criteria  $c_2, c_3$  and  $c_4$  are 0.333, 0.167 and 0 respectively, and also exhibit a gradation relationship.

The numerical examples show that the additive AHP model has the advantages of being easily understood and easy applied.

## 6. Fuzzy Additive Scale of AHP Model

As pair comparisons sometimes containing inevitably fuzziness in human judgment and preference, particularly for intangible items such as degree of customer satisfaction, quality of product (service), quality of input resources, input data only providing a crisp number do not satisfy real needs. This restriction will significantly decreases the practical flexibility of the ratio and additive AHP models. Thereby, a numerical papers had discussed fuzzy ratio AHP, those refer to Chang and Lee [41], Zhu, Jing and Chang [42], Mikhailov [43] and Wang and Elhag [44]. This study assumes that the value of  $\tilde{d}_{i,j}$  of additive AHP model is a fuzzy member. Additionally, let  $\tilde{D}_{ij}$  be the fuzzy number defined on the universal  $X_{ij}$ . Thus the fuzzy membership function  $u_{\tilde{D}_{ij}}$  has the form: (Kaufmann [17], Kandal [16], Kickert [18], Klir & Folger [19] and Zimmermann [33])

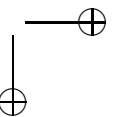
$$u_{\tilde{D}_{ij}} : X_{ij} \rightarrow [0, 1], \quad \text{for } i, j = 1, \dots, n.$$

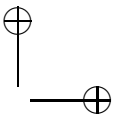
Furthermore, we assumed that  $u_{\tilde{D}_{ij}}$  is a fuzzy membership function, and thus the  $\alpha$ -cut of fuzzy set  $\tilde{D}_{ij}$  is denoted as

$$\tilde{D}_{ij}^\alpha = \left\{ \tilde{d}_{ij}^\alpha \in X_{ij} / u_{\tilde{D}_{ij}}(\tilde{d}_{ij}) \geq \alpha \right\} = [d_{ij}^{\alpha,L}, d_{ij}^{\alpha,R}]$$

where superscript  $L$  indicates the extreme left value and  $R$  indicates the extreme right value of the previously defined universal set. Let  $\tilde{W}_i$  be the fuzzy number of the estimated weight of o criterion  $c_i$ , which is defined on the universal  $Y_i$ . The fuzzy membership function  $u_{\tilde{W}_i}$  then has the form:

$$u_{\tilde{W}_i} : Y_i \rightarrow [0, 1], \quad \text{for } i = 1, \dots, n.$$





Hence, the  $\alpha$ -cut of fuzzy set  $\widetilde{W}_i$  is as follows:

$$\widetilde{W}_i^\alpha = \left\{ \widetilde{w}_i^\alpha \in Y_i / u_{\widetilde{w}_i}(w_i) \geq \alpha \right\} = [w_i^{\alpha,L}, w_i^{\alpha,R}]$$

As  $\hat{w}'_k = (10(n-1) + \sum_{j=1}^n d_{k,j}) / 10n(n-1)$  and  $d_{i,j}^{\alpha,L} \leq d_{i,j}^\alpha \leq d_{i,j}^{\alpha,R}$ , for each  $\alpha$ -cut, the upper bound ( $\hat{w}'_k{}^{\alpha,R}$ ) and lower bound ( $\hat{w}'_k{}^{\alpha,L}$ ) of normalized weight  $\hat{w}'_k$  are obtained using the following equations:

$$\begin{aligned} \hat{w}'_k{}^{\alpha,R} &= \left( 10(n-1) + \sum_{j=1}^n d_{k,j}^{\alpha,R} \right) / 10n(n-1) \\ \hat{w}'_k{}^{\alpha,L} &= \left( 10(n-1) + \sum_{j=1}^n d_{k,j}^{\alpha,L} \right) / 10n(n-1) \end{aligned}$$

### Example 2.

To illustrate the fuzzy additive AHP model, a simple problem with three criteria to be considered. Suppose the  $\widetilde{d}_{i,j}$  is a triangular fuzzy member, the interval at a specific  $\alpha$ -cut is denoted as  $\widetilde{D}^\alpha = \{\widetilde{d}_{i,j}^\alpha\} = \{[d_{i,j}^{\alpha,L}, d_{i,j}^{\alpha,R}]\}$ . The vertex point of the triangular fuzzy number is the average of  $d_{i,j}^{\alpha,L}$  and  $d_{i,j}^{\alpha,R}$ . The following is the interval matrix for  $\widetilde{D}^0$ ,  $i, j = 1, 2, 3$ :

$$\widetilde{D}_{ij}^0 = \left\{ [d_{ij}^{0,L}, d_{ij}^{0,R}] \right\} = \begin{bmatrix} (0, 0) & (2, 4) & (-1, 3) \\ (-4, -2) & (0, 0) & (-6, -4) \\ (-3, 1) & (4, 6) & (0, 0) \end{bmatrix}.$$

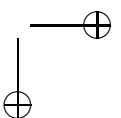
The upper and lower bound of normalized weight  $\hat{w}'_1$ ,  $\hat{w}'_2$  and  $\hat{w}'_3$  are calculated as follows:

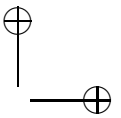
$$\begin{aligned} [\hat{w}'_1{}^{0,L}, \hat{w}'_1{}^{0,R}] &= \left[ \frac{10 \times 2 + 2 + (-1)}{10 \times 3 \times 2}, \frac{10 \times 2 + 4 + 3}{10 \times 3 \times 2} \right] = [0.3000, 0.45000] \\ [\hat{w}'_2{}^{0,L}, \hat{w}'_2{}^{0,R}] &= \left[ \frac{10 \times 2 + (-4) + (-6)}{10 \times 3 \times 2}, \frac{10 \times 2 + (-2) + (-4)}{10 \times 3 \times 2} \right] = [0.1667, 0.23333] \\ [\hat{w}'_3{}^{0,L}, \hat{w}'_3{}^{0,R}] &= \left[ \frac{10 \times 2 + (-3) + 4}{10 \times 3 \times 2}, \frac{10 \times 2 + 1 + 6}{10 \times 3 \times 2} \right] = [0.3500, 0.45000] \end{aligned}$$

Similarly, the  $[\hat{w}'_k{}^{\alpha,L}, \hat{w}'_k{}^{\alpha,R}]$  for  $k = 1, 2$  and  $3$  at  $\alpha = 0, 1/3, 2/3$  and  $1$  can be obtained. The results are listed in Table 2.

**Table 2.** The normalized weight for different  $\alpha$ -cuts in the fuzzy additive AHP model.

	$\alpha = 0$	$\alpha = 1/3$	$\alpha = 2/3$	$\alpha = 1$
$[\hat{w}'_1{}^{\alpha,L}, \hat{w}'_1{}^{\alpha,R}]$	[0.3000, 0.4500]	[0.3667, 0.4333]	[0.3833, 0.4167]	[0.4000, 0.4000]
$[\hat{w}'_2{}^{\alpha,L}, \hat{w}'_2{}^{\alpha,R}]$	[0.1667, 0.2333]	[0.1778, 0.2222]	[0.1889, 0.2111]	[0.2000, 0.2000]
$[\hat{w}'_3{}^{\alpha,L}, \hat{w}'_3{}^{\alpha,R}]$	[0.3500, 0.4500]	[0.3667, 0.4333]	[0.3833, 0.4167]	[0.4000, 0.4000]





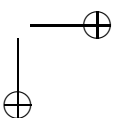
## 7. Conclusion

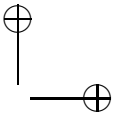
The main objective of this study is not to criticize the ratio scale adopted in the AHP, but rather to provide an alternative scale for use, the additive scale, that can help the AHP to fit a decision problem using linear preference comparison. The additive AHP model is based on theoretical deduction, which is readily understood, easily implemented, and capable of producing results that agree with expectations. The advantages of the additive AHP model are that criteria weights obtained are steady and effectively reflect the intensity of perception, the scale transitivity is more moderate than the conventional ratio scale of AHP model, and its CI is invariant to the scale multiplier used. Table 3 compares the ratio and additive model of AHP.

**Table 3.** Comparison of ratio and additive AHP models.

	Ratio model of AHP	additive model of AHP
<b>Basic feature</b>		
Scale assumption	Ratio ( $w_i/w_j$ ) for all $i, j$	Interval ( $w_i/w_j$ ) for all $i, j$
Axiomatic foundation	- reciprocal	- subtraction
	- homogeneity	- homogeneity
	- independence	- independence
	- expectation	- expectation
Scale gradation	1, 2, 3, ..., 9	0, 1, 2, 3, ..., 10
<b>Solution process</b>		
Solution method	Eigenvalue method	Least squares method
	Geometric Mean,	
	Logarithmic Least Squares,	
	Least Absolute Values,	
	Chi Squares Method, etc.	
pairwise comparison	$n(n-1)/2$	$n(n-1)/2$
<b>Consistency index</b>		
method	$CI = (\lambda_{\max} - n)/(n-1)$	product-moment correlation
scope	0 to 1	-1 to 1
invariant to scale multiplier	no	yes
<b>Scale transitivity</b>	ratio transitivity ( $v_k v_j = v_{k \times j}$ )	additive transitivity ( $v_k v_j = v_{k+j}$ )

Numerous issues of the additive AHP model deserve further exploitation, such as comparisons with various ratio scale models in the AHP, MAUT, SMART and other decision analysis methods, the development of various fuzzy models for additive AHP model, and the extension of mathematical efforts. Additionally, although this study demonstrates that the additive model of AHP accurately portrays linear preference, the additive and ratio scale are two models of mankind's perception to reflect the intensity of the pairwise comparison. Thus, even someone makes a ratio comparison, it is inevitably confounding additive preference perception. Based on this explanation, the ideal input comparison data is composed of ratio and additive effects. This assumption can be





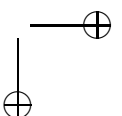
symbolically represented as follows:

$$a_{ij} = \alpha \left( \frac{w_i}{w_j} \right) + \beta(w_i - w_j) + \varepsilon_{ij}, \quad \text{for } i, j = 1, \dots, n.$$

where  $a_{ij}$  is the cell in AHP paired criteria comparison matrix, which is generated by comparing the perferability between criteria (objective)  $c_i$  and  $c_j$ ; the components of  $(w_i/w_j)$  and  $(w_i - w_j)$  are ratio scale effect and additive scale effect, which is a combination of parameters  $\alpha$  and  $\beta$ ,  $\alpha + \beta = 1$ ,  $\alpha \geq 0$  and  $\beta \geq 0$ ,  $\varepsilon_{ij}$  is judgment error. Hence, separating the ratio and additive scale component from a pairwise comparison matrix assigned by DM, and then re-estimate the relative weights of criteria by considering these both these effects simultaneously is an issue that further deserves exploration.

## References

- [1] Alkahtani, A. M. S., Woodward, M. E. and Al-Begain, K., *Prioritised best effort routing with four quality of service metrics applying the concept of the analytic hierarchy process*, Comput and Opl Res., Vol. 33, pp.559-580, 2006.
- [2] Badri, M. A., Abdulla, M. and Al-Madani, A., *Information technology center service quality: Assessment and application of SERVQUAL*, J. Qual. Reliab Management, Vol. 22, pp.819-848, 2005.
- [3] Belton, V. and Gear, A. E., *On a shortcoming of Saaty' method of analytical hierarchies*, Omega, Vol. 11, pp.227-230, 1983.
- [4] Belton, V. and Gear, A. E. *The legitimacy of rank reversal- a comment*, Omega, Vol. 13, pp.143-144, 1985.
- [5] Chow, C. C. and Luk, P., *A strategic service quality approach using analytic hierarchy process*, Manag Serv Qual, Vol. 15, pp.278-289, 2005.
- [6] Dodd, F. J., Donegan, H. A and McMaster, T. B. M., *Scale horizons in analytic hierarchies*, J. Multi-criteria Decis Anal., Vol. 4, pp.177-188, 1995.
- [7] Donegan, H. A., Dodd, F. J. and McMaster, T. B. M., *A new approach to AHP decision-making*, Statistician, Vol. 41, pp.295-302, 1992.
- [8] Fichtner, J., *On deriving priority vectors from matrices of pairwise comparisons*, Socio-Economic Planning Sci., Vol. 20, pp.341-345, 1986.
- [9] Forgionne, G. A., Kohli, R. and Jennings, D., *An AHP analysis of quality in AI and DSS journals*, Omega, Vol. 30, pp.171-183, 2002.
- [10] Forman, E. H. and Gass, S. L., *The analytic hierarchy process-an exposition*, Opns Res., Vol. 49, pp.469-486, 2001.
- [11] Foster, B. and Al-Dubaibi, A., *An axiomatic approach to pairwise comparisons*. Working paper. Department of Mathematics and Statistics Brunel University, 1995.
- [12] Harker, P. T. and Vargas, L. G., *The theory of ratio scale estimation: Satty analytic hierarchy process*, Mngt Sci., Vol. 33, pp.1383-1403, 1987.
- [13] Ji, P. and Jiang, R., *Scale transitivity in the AHP*, J. Opl. Res. Soc., Vol. 54, pp.896-905, 2003.
- [14] Kahraman, C., Cebeci, U. and Ruan, D., *Multi-attribute comparison of catering service companies using fuzzy AHP: The case of Turkey*, J. Prod. Econs., Vol. 87, pp.171-184, 2004.
- [15] Kahraman, C., Ertay, T. and Buyukozkan, G., *A fuzzy optimization model for QFD planning process using analytic network approach*, Eur. J. Opl. Res. Vol. 171, pp.390-411, 2006.
- [16] Kandal, A., *Fuzzy Mathematical Techniques with Application*, Addison- Wesley. Mass., 1986.
- [17] Kaufmann, A., *Introduction to the theory of fuzzy subsets*, Academic Press, New York, 1975.
- [18] Kickert, W. J. M., *Fuzzy Theories on Decision Making*, Martines Nijhoff, Boston, 1987.
- [19] Klir, G. J. and Folger, T. A., *Fuzzy Sets, Uncertainty and Information*, Prentice- Hall, London, 1988.

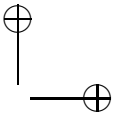




- [20] Lootsma, F. A., *Scale sensitivity in the multiplicative AHP and SMART*, J. Multi-criteria Decis Anal., Vol. 2, pp.87-110, 1993.
- [21] Lootsma, F. A., *Multi-criteria Decision Analysis via Ratio and Difference Judgment*, Kluwer Academic Publishers: Dordrecht, 1999.
- [22] Ma, D. and Zheng, X., 9/9-9/1 Scale Method of AHP. Proceedings of the second International Symposium on the AHP, Vol. 1. University of Pittsburgh, Pittsburgh, PA, pp.197-202, 1991.
- [23] Saaty, T. L. *A scaling method for priorities in hierarchical structures*, J. Math. Psychol, Vol. 15, pp.234-281, 1977.
- [24] Saaty, T. L., *The Analytic Hierarchy Process*. McGraw-Hill, New York, 1980.
- [25] Saaty, T. L., *Axiomatic foundation of the analytic hierarchy process*, Management Sci., Vol. 32, pp.841-855, 1986.
- [26] Saaty, T. L., *Highlights and critical points in the theory and application of the analytic hierarchy process*, Eur. J. Opl. Res., Vol. 74, pp.426-447, 1994.
- [27] Salo, A. A. and Hamalainen, R. P., *On the measurement of preferences in the analytic hierarchy process*, J. Multi-criteria Decis Anal., Vol. 6, pp.309-319, 1997.
- [28] Sampson, S. E. and Showalter, M. J., *The performance-importance response function: Observations and implications*, J. Service Industries I, Vol. 19, pp.1-25, 1999.
- [29] Tsaour, S. H. and Lin, Y. C., *Promoting service quality in tourist hotels: The role of HRM practices and service behavior*, Tourism Management, Vol. 25, pp.471-481, 2004.
- [30] Torgerson, W. S., *Distances and Ratios in Psych-Physical Scaling*, Acta Psychological, Vol. XIX, pp.201-205, 1961.
- [31] Vargas, L. G., *An overview of the analytic hierarchy process and its applications*, Eur. J. Opl. Res., Vol. 48, pp.2-8, 1990.
- [32] Zimmermann, H. J., *Fuzzy set theory - and its applications*, Kluwer-Nijhoff, Boston, 1985.
- [33] Crawford, G. and Williams, C., *A note on the analysis of subjective judgment matrices*, J. of Mathematical Psychology, Vol. 29, pp.387-405, 1985.
- [34] Golany, B., Kress, M.(1993). A multicriteria evaluation of methods for obtaining weights from ratio-scale matrices, Eur. J. Opl Res, 69: 210-220
- [35] Jensen, R. E., Comparison of Eigenvector, Least squares, Chi square and Logarithmic least square methods of scaling a reciprocal matrix, Working Paper 153, 1983. <http://www.trinity.edu/rjensen/127wp/127wp.htm>
- [36] Saaty, T. L. and Vargas, L. G., *Comparison of eigenvalues, logarithmic least squares and least squares methods in estimating ratios*, Mathematical Modeling, Vol. 5, pp.309-324, 1984.
- [37] Takeda, E., Cogger, K. O. and Yu, P. L., *Estimating criterion weights using eigenvectors: A comparative study*, Eur. J. Opl Res., Vol. 29, pp.360-369, 1987.
- [38] Ali, I., Cook, W. D. and Kress, M., *Ordinal ranking and intensity of preference: a linear programming approach*, Management Science, Vol. 32, pp.1642-1647, 1986.
- [39] Hochbaum, D.S., Walter, A. and A. Levin (2006). Methodologies and algorithms for group rankings decision, <http://pluto.mscc.huji.ac.il/~levinas/nsf.pdf>.
- [40] Chang, P. T. and Lee, E. S., *The estimation of normalized fuzzy weights*, Computers & Mathematics with Applications, Vol. 29, pp.21-42, 1995.
- [41] Zhu, K. J., Jing, Y. and Chang, D. Y., *A discussion on Extent Analysis Method and applications of fuzzy AHP*, European Journal of Operational Research, Vol. 116, pp.450-456, 1999.
- [42] Mikhailov, L., *Group prioritization in the AHP by fuzzy preference programming method*, Computers & Operations Research, Vol. 3, pp.293-301, 2004.
- [43] Wang, Y. M. and Elhag, T. M. S., *On the normalization of interval and fuzzy weights*, Fuzzy Sets and Systems, Vol. 157, pp.2456-2471, 2006.

### Authors' Information

Yuh-Yuan Guh is currently a Professor in the Department of Business Administration, Chung Yuan Christian University, Taiwan, R.O.C.. He received his Ph.D. in Industrial Engineering from National



88 International Journal of Information and Management Sciences, Vol. 20, No. 1, March, 2009

Tsing-Hua University. His research interests are Operation research, Service system design, Service science, Decision management.

Department of Business Administration, Chung Yuan Christian University, Chung-Li 32023, Taiwan, R.O.C.

E-mail: yuhyuan@cycu.edu.tw TEL: +886-3-265-2500.

Rung-Wei Po is currently a doctoral student in the Institute of Technology Management of National Tsing-Hua University. She is majoring in Knowledge base service system design, Service science, Decision management.

Institute of Technology Management, National Tsing-Hua University, Hsinchu 30013, Taiwan, R.O.C.

E-mail: rd9673802@oz.nthu.edu.tw TEL: +886-3-437-7559.

Kuo-Ren Lou is currently an associate professor in the Department of Management Sciences & Decision Making at Tamkang University in Taiwan. He received his Ph.D. degree in the Department of Statistics from the University of Connecticut, U.S.A.

Department of Management Sciences & Decision Making, Tamkang University, Tamsui, Taipei, 25137, Taiwan, R.O.C.

E-mail: 109880@mail.tku.edu.tw TEL: +886-2-2621-5656 ext 2845

